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# Mirror symmetry and generalized complex manifolds Part II. Integrability and the transform for torus bundles

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**Abstract**

In this paper we continue the development of a relative version of T-duality in generalized complex geometry which we propose as a manifestation of mirror symmetry. We discuss the integrability of the transform from Part I in terms of data on the base manifold. We work with semi-flat generalized complex structures on real $n$-torus bundles with section over an $n$-dimensional base and use the transform on vector bundles developed in Part I of this paper to discuss the bijective correspondence between semi-flat generalized complex structures on pairs of dual torus bundles. We give interpretations of these results in terms of relationships between the cohomology of torus bundles and their duals. We comment on the ways in which our results generalize some well established aspects of mirror symmetry. Along the way, we give methods of constructing generalized complex structures on the total spaces of the bundles we consider.


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## 1. Introduction

In the associated paper, Part I: The Transform on Vector Bundles, Spinors, and Branes [1], we gave transformation rules for generalized almost complex structures on vector bundles, including various assumptions and compatibility conditions. We have also commented on the mirror transformation on spinors and branes, as well as a relationship to Dirac geometries on the base manifold. Furthermore, we have examined the geometry of a pair of transverse foliations, and the compatibilities with generalized complex and generalized Kähler structures. We have proposed this transformation as a very simple case of mirror symmetry. In this paper, we continue the analysis, focusing on the integrability conditions, and the new features that arise in the case of torus bundles. Finally, we work out some more explicit details in certain examples. In Part I, we have included a more complete introduction, with background material and additional references.

In this paper, we relate the integrability of *semi-flat* (see Definition 2.2) generalized almost complex structures on torus and vector bundles to data which lives only on the base manifold. We show that a semi-flat generalized almost complex structure is integrable if and only its mirror structure is integrable.

Using a natural connection on a torus bundle $Z \to M$ with zero section $s$, we will construct semi-flat generalized complex structures $\mathcal{J}$ on $Z$ from generalized almost complex structures $\underline{\mathcal{J}}$ on the vector bundle $s^*T_{Z/M} \oplus T_M$. The definition of semi-flat includes the condition that

$$\underline{\mathcal{J}}(s^*T_{Z/M} \oplus s^*T_{Z/M}^\vee) = T_M \oplus T_M^\vee.$$

Then we have the following two results.

**Theorem 1.1** (3.4). *A semi-flat generalized almost complex structure $\mathcal{J}$ on a torus bundle $Z \to M$ with zero section $s$ is integrable if and only if*

$$[\underline{\mathcal{J}}(\mathcal{S} \oplus \mathcal{S}^\vee), \underline{\mathcal{J}}(\mathcal{S} \oplus \mathcal{S}^\vee)] = 0,$$

*where $\mathcal{S}$ is the sheaf of flat sections of $s^*T_{Z/M}$.*

**Corollary 1.2** (3.5). *A semi-flat generalized almost complex structure $\mathcal{J}$ on a torus bundle $Z \to M$ is integrable if and only if its mirror structure $\hat{\mathcal{J}}$ on the dual torus bundle $\hat{Z} \to M$ is integrable.*

These statements set the stage for understanding mirror symmetry and the mirror transform of D-branes in generalized Calabi–Yau geometry. Our results are a direct generalization of the setup employed by Arinkin and Polishchuk [32] in ordinary mirror symmetry. Explicit examples of this fact can be found in Section 5.

We relate this transformation of geometric structures to a purely topological map on differential forms which descends to a map from the de Rham cohomology of $Z$ to the de Rham cohomology of $\hat{Z}$. In particular, the map on differential forms exchanges the pure spinors associated to the generalized complex structure on $Z$ with the ones associated to the mirror generalized complex structure on $\hat{Z}$. This type of transformation was also discussed in [14].

Throughout the paper we comment on how our results relate to some of the well established results and conjectures of mirror symmetry [32,34,26,31,25] and also what they say in regards to the new developments in generalized Kähler geometry [16] and the relationships between generalized complex geometry and string theory [22,36,16] which have appeared recently. As mentioned in [22] we may interpret these dualities as being a generalization of the duality between the A- and B-model in topological string theory. In the generalized Kähler case, they can be interpreted as dualities of supersymmetric nonlinear sigma models [15].

### 1.1. Integrability

Let $M$ be a generalized almost complex manifold. The *Courant bracket* ([10], p. 645) is defined on sections of $(T_M \oplus T_M^\vee) \otimes \mathbb{C}$ by

$$[X+\xi, Y+\eta] = [X,Y] + \mathcal{L}_X\eta - \mathcal{L}_Y\xi + \frac{1}{2}\,\mathrm{d}(\iota_Y\xi - \iota_X\eta),$$

or equivalently

$$[X+\xi, Y+\eta] = [X,Y] + \iota_X\,\mathrm{d}\,\eta + \frac{1}{2}\,\mathrm{d}\iota_X\eta - \iota_Y\,\mathrm{d}\xi - \frac{1}{2}\iota_Y\xi.$$

**Definition 1.3** (cf. [16,18,19]). Let $M$ be a real manifold equipped with a generalized almost complex structure defined by $E \subseteq (T_M \oplus T_M^\vee) \otimes \mathbb{C}$. We say that $E$ is *integrable* if the sheaf of sections of $E$ is closed under the Courant bracket. If that is the case, we also say that $E$ is a *generalized complex structure* on $M$, and that $M$ is a *generalized complex manifold*.

**Remark 1.4** (cf. [10,19]). As we noted in [2], the integrability condition for a generalized almost complex structure $\mathcal{J}$ is equivalent [2] to the vanishing of the *Courant–Nijenhuis* tensor:

$$N_{\mathcal{J}}(X,Y) = [\mathcal{J}X, \mathcal{J}Y] - \mathcal{J}[\mathcal{J}X, Y] - \mathcal{J}[X, \mathcal{J}Y] - [X,Y],$$

where $X,Y$ are sections of $T_M \oplus T_M^\vee$.

Integrability can also be expressed in terms of spinors [19,16,40]. If $L \subseteq \bigwedge^\bullet T_M^\vee \otimes \mathbb{C}$ is the line bundle of spinors associated to a generalized almost complex structure $\mathcal{J}$ on a manifold $M$ then $\mathcal{J}$ is integrable if and only if all sections $\phi$ of $L$ satisfy

$$\mathrm{d}\phi = \iota_v\phi + \alpha \wedge \phi$$

for some section $v+\alpha$ of $(T_M \oplus T_M^\vee) \otimes \mathbb{C}$.

**Example 1.5** (cf. [19,16]). In the case that a generalized almost complex structure comes from an almost complex structure, it will be integrable if and only if the almost complex structure is integrable, giving a complex structure to the manifold. In the case that a generalized almost complex structure comes from a non-degenerate differential two-form (almost symplectic structure), it will be integrable if and only if the form is closed, i.e. gives a sympletic structure to the manifold.

General $B$-field and $\beta$-field transformations need not preserve integrability. However, [18], a *closed* two-form $B$ acts on generalized complex structures on $M$ in the same way as described in Part I of this paper under the section on notations, conventions, and basic definitions [1]. In fact, a $B$-transform by a two-form on $M$ is an automorphism of the Courant bracket if and only if the two-form is closed [19]. Note further that a $B$-field transform of a particular generalized complex structure can be integrable even if the two-form is not closed. In fact, for any specific generalized complex manifold $(M, \mathcal{J})$, one can write down explicitly the conditions that need to be satisfied by a two form, $B$ or a bi-vector field $\beta$ in order for the $B$-field or $\beta$-field transform of $(M, \mathcal{J})$ to be integrable. We will study examples of this phenomenon in Section 5.

## 2. The question of integrability

The purpose of this section is to express the integrability of $\nabla$-lifted, adapted generalized almost complex structures $\mathcal{J}$ on the total space of vector bundles in terms of data on the base manifold $M$. We do this only in the case where $\nabla$ is flat (in which case we call the structures $\mathcal{J}$ *semi-flat*). Once we do this it will be clear that $\mathcal{J}$ on $X = \mathrm{tot}(V)$ is integrable if and only if the mirror structure $\hat{\mathcal{J}}$ on $\hat{X} = \mathrm{tot}(V^\vee)$ is integrable. For all notation used in this section, see Section 4 of Part I [1]. In particular, we shall use the maps:

$$\underline{\mathcal{J}} = \begin{pmatrix} 0 & \mathcal{J}_{12} & 0 & \mathcal{J}_{22} \\ \mathcal{J}_{13} & 0 & -\mathcal{J}_{22}^\vee & 0 \\ 0 & \mathcal{J}_{31} & 0 & -\mathcal{J}_{13}^\vee \\ -\mathcal{J}_{31}^\vee & 0 & -\mathcal{J}_{12}^\vee & 0 \end{pmatrix}, \quad \underline{\mathcal{J}} \in GL(V \oplus T_M \oplus V^\vee \oplus T_M^\vee) \tag{2.1}$$

and

$$\underline{\hat{\mathcal{J}}} = \begin{pmatrix} 0 & \mathcal{J}_{31} & 0 & -\mathcal{J}_{13}^\vee \\ -\mathcal{J}_{22}^\vee & 0 & \mathcal{J}_{13} & 0 \\ 0 & \mathcal{J}_{12} & 0 & \mathcal{J}_{22} \\ -\mathcal{J}_{12}^\vee & 0 & -\mathcal{J}_{31}^\vee & 0 \end{pmatrix}, \quad \underline{\hat{\mathcal{J}}} \in GL(V^\vee \oplus T_M \oplus V \oplus T_M^\vee). \tag{2.2}$$

The relationship of these maps to $\mathcal{J}$ and $\hat{\mathcal{J}}$ was worked out in great detail in Part I [1]. Recall that the choice of a connection $\nabla$ gives rise to a splitting $(D, \alpha)$:

$$0 \longrightarrow \pi^* V \underset{D}{\overset{j}{\rightleftarrows}} T_X \underset{\alpha}{\overset{d\pi}{\rightleftarrows}} \pi^* T_M \longrightarrow 0,$$

of the tangent sequence of $\pi : X \to M$, and a dual splitting of the tangent sequence of $\hat{\pi} : \hat{X} \to M$.

Now if that $\nabla$ *is flat* it is known [24] that we may find in a neighborhood of any point of $M$ a frame, $\{e_i\}$ such that $\nabla e_i = 0$. We will call $\{e_i\}$ a *flat frame*. Given a flat frame, along with the corresponding vertical coordinates $\{\xi_i\}$ we have that for any choice of coordinates $x_i$ on the base, the functions $\xi_i$ together with $y_i = x_i \circ \pi$ form a coordinate system on $X$ and $\alpha(\pi^* \partial/\partial x_i) = (1 - j \circ D)\partial/\partial y_i = \partial/\partial y_i$ follows from the expression in this frame (see the analysis in Part I of this paper in Section 4 [1]) for $D$. We define a frames $f_i$ for $\pi^* V$ and $f^i$ for $\pi^* V^\vee$ by using the pullbacks $f_i = \pi^* e_i$ and $f^i = \pi^* e^i$, where $\{e^i\}$ is a dual frame to $\{e_i\}$.

**Remark 2.1.** Notice that $\nabla$ is flat if and only if the image of $\alpha$ is involute. Hence, in this case we have a horizontal foliation instead of just a horizontal distribution. We considered the geometry of a pair of transverse foliations and its interaction with a generalized complex structure in Part I, Section 7.

**Definition 2.2.** If $\nabla$ is a *flat* connection on a rank $n$ vector bundle $V$ over a real $n$-manifold then a $\nabla$-*semi-flat* generalized almost complex structure on $X = \text{tot}(V)$ is an adapted, $\nabla$-lifted (see Part I, Definitions 4.1 and 4.2 [1]) generalized almost complex structure.

Let $\mathcal{S}$ be the sub-sheaf of flat sections of $V \oplus V^\vee$. Consider the isomorphism of vector bundles:

$$\mathcal{M} : V \oplus V^\vee \to T_M \oplus T_M^\vee,$$

$$\mathcal{M} = \begin{pmatrix} \mathcal{J}_{13} & \mathcal{J}_{22}^\vee \\ -\mathcal{J}_{31}^\vee & \mathcal{J}_{12}^\vee \end{pmatrix}, \qquad \mathcal{M}^{-1} = \begin{pmatrix} -\mathcal{J}_{12} & -\mathcal{J}_{22} \\ -\mathcal{J}_{31} & \mathcal{J}_{13}^\vee \end{pmatrix}. \tag{2.3}$$

With this notation we have the following theorem.

**Theorem 2.3.** *If $V$ is a vector bundle on a manifold $M$, then a semi-flat generalized almost complex structure $\mathcal{J} = F^{-1}(\pi^* \underline{\mathcal{J}})F$ on $X = \text{tot}(V)$ is integrable if and only if all pairwise Courant brackets of sections of the sheaf $\mathcal{M}(\mathcal{S})$ vanish.*

Notice that this condition is expressed entirely in terms of data on the base manifold $M$. Furthermore, we will see that this theorem implies the following corollary.

**Corollary 2.4.** *The generalized almost complex structure $\mathcal{J} = F^{-1}(\pi^* \underline{\mathcal{J}})F$ on $X = \text{tot}(V)$ is integrable if and only if the generalized almost complex structure $\hat{\mathcal{J}} = \hat{F}^{-1}(\hat{\pi}^* \underline{\hat{\mathcal{J}}})\hat{F}$ on $\hat{X} = \text{tot}(V^\vee)$ is integrable.*

**Remark 2.5.** In other words the mirror symmetry transformation is a bijective correspondence between $\nabla$-semi-flat generalized complex structures on $X$ and $\nabla^\vee$-semi-flat generalized complex structures on $\hat{X}$.

**Example 2.6.** If $\nabla$ is any flat, torsion-free connection on $T_M$, we can put a canonical complex structure on $\text{tot}(T_M)$. See Section 5 for more details. This construction was first done in [11]. It is easy to see that the mirror structure is *always* the canonical symplectic structure on $\mathrm{tot}(T_M^\vee)$.

**Proof of Theorem 2.3.** Let us analyze the condition that the (+i) eigenbundle $E$ be involute. The bundle $E$ is the graph of the isomorphism:

$$-\mathrm{i}\mathcal{J}|_{\mathrm{image}(j\oplus D^\vee)\otimes\mathbb{C}} : \mathrm{image}(j \oplus D^\vee) \otimes \mathbb{C} \to \mathrm{image}(\alpha \oplus \mathrm{d}\pi^\vee) \otimes \mathbb{C}.$$

It suffices to analyze involutivity it locally on the base manifold. Note that in the local frame and coordinates which we have chosen, we have the following formulae:

$$j(f_i) = \frac{\partial}{\partial \xi_i}, \qquad D\left(\frac{\partial}{\partial \xi_i}\right) = f_i, \qquad D\left(\frac{\partial}{\partial y_i}\right) = 0, \qquad \alpha\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial y_i},$$
$$\mathrm{d}\pi\left(\frac{\partial}{\partial y_i}\right) = \pi^*\frac{\partial}{\partial x_i}, \qquad \mathrm{d}\pi\left(\frac{\partial}{\partial \xi_i}\right) = 0, \qquad D^\vee(f^i) = \mathrm{d}\xi_i,$$
$$\alpha^\vee(\mathrm{d}y_i) = \pi^*\mathrm{d}x_i, \qquad \alpha^\vee(\mathrm{d}\xi_i) = 0, \qquad (\mathrm{d}\pi)^\vee(\pi^*\,\mathrm{d}x_i) = \mathrm{d}y_i,$$
$$j^\vee(\mathrm{d}\xi_i) = f^i, \qquad j^\vee(\mathrm{d}y_i) = 0.$$

Furthermore, an isotropic sub-bundle of $(T_X \oplus T_X^\vee) \otimes \mathbb{C}$ is involute if and only if it has a basis of sections whose pairwise Courant brackets are themselves sections of the original bundle. This follows immediately from the Leibniz property of the Courant bracket, see e.g. [40,10]. This property says that

$$[v_1 + \alpha_1, f(v_2 + \alpha_2)] = f[v_1 + \alpha_1, v_2 + \alpha_2] + v_1(f)(v_2 + \alpha_2) + \langle v_1 + \alpha_1 v_2 + \alpha_2 \mathrm{d}f\rangle \tag{2.4}$$

for all vector fields $v_1$ and $v_2$, one-forms $\alpha_1$ and $\alpha_2$ and functions $f$. Let $U$ is the coordinate neighborhood of the base. We will analyze involutivity in $\pi^{-1}(U)$, using the coordinate system and frame described above. Involutivity of $E$ is equivalent to the condition that $[a_i, a_j]$, $[a_i, b_j]$, and $[b_i, b_j]$ are all sections of $E$ where

$$a_i = j(f_i) - \mathrm{i}\mathcal{J}(j(f_i))$$

and

$$b_i = D^\vee(f^i) - \mathrm{i}\mathcal{J}(D^\vee(f^i)).$$

Using the special form of $\mathcal{J}$ we have

$$a_i = \frac{\partial}{\partial \xi_i} - \mathrm{i}\alpha\pi^*(\mathcal{J}_{13}e_i) + \mathrm{i}(\mathrm{d}\pi)^\vee\pi^*(\mathcal{J}_{31}^\vee e_i)$$

and

$$b_i = \mathrm{d}\xi_i + \mathrm{i}\alpha\pi^*(\mathcal{J}_{22}^\vee e^i) + \mathrm{i}\mathrm{d}\pi^\vee\pi^*(\mathcal{J}_{12}^\vee e^i).$$

Hence, we have that

$$
\begin{aligned}
[a_i, a_j] &= \left[\frac{\partial}{\partial \xi_i} - \mathrm{i}\alpha\pi^*(\mathcal{J}_{13}e_i) + \mathrm{i}(\mathrm{d}\pi)^\vee\pi^*(\mathcal{J}_{31}^\vee e_i), \frac{\partial}{\partial \xi_j}\right.\\
&\quad \left. - \mathrm{i}\alpha\pi^*(\mathcal{J}_{13}e_j) + \mathrm{i}(\mathrm{d}\pi)^\vee\pi^*(\mathcal{J}_{31}^\vee e_j)\right]\\
&= \left[\frac{\partial}{\partial \xi_i} - \mathrm{i}\alpha\pi^*(\mathcal{J}_{13}e_i), \frac{\partial}{\partial \xi_j} - \mathrm{i}\alpha\pi^*(\mathcal{J}_{13}e_j)\right]\\
&\quad + \iota_{\partial/\partial\xi_i - \mathrm{i}\alpha\pi^*(\mathcal{J}_{13}e_i)}\mathrm{d}\mathrm{i}(\mathrm{d}\pi)^\vee\pi^*(\mathcal{J}_{31}^\vee e_j)\\
&\quad - \iota_{\partial/\partial\xi_j - \mathrm{i}\alpha\pi^*(\mathcal{J}_{13}e_j)}\mathrm{d}\mathrm{i}(\mathrm{d}\pi)^\vee\pi^*(\mathcal{J}_{31}^\vee e_i)\\
&\quad + \frac{1}{2}\mathrm{d}\iota_{\partial/\partial\xi_i - \mathrm{i}\alpha\pi^*(\mathcal{J}_{13}e_i)}\mathrm{i}(\mathrm{d}\pi)^\vee\pi^*(\mathcal{J}_{31}^\vee e_j)\\
&\quad - \frac{1}{2}\mathrm{d}\iota_{\partial/\partial\xi_j - \mathrm{i}\alpha\pi^*(\mathcal{J}_{13}e_j)}\mathrm{i}(\mathrm{d}\pi)^\vee\pi^*(\mathcal{J}_{31}^\vee e_i)\\
&= -[\alpha\pi^*(\mathcal{J}_{13}e_i), \alpha\pi^*(\mathcal{J}_{13}e_j)] + \iota_{\alpha\pi^*(\mathcal{J}_{13}e_i)}\mathrm{d}(\mathrm{d}\pi)^\vee\pi^*(\mathcal{J}_{31}^\vee e_j)\\
&\quad - \iota_{\alpha\pi^*(\mathcal{J}_{13}e_j)}\mathrm{d}(\mathrm{d}\pi)^\vee\pi^*(\mathcal{J}_{31}^\vee e_i) + \frac{1}{2}\mathrm{d}\iota_{\alpha\pi^*(\mathcal{J}_{13}e_i)}(\mathrm{d}\pi)^\vee\pi^*(\mathcal{J}_{31}^\vee e_j)\\
&\quad - \frac{1}{2}\mathrm{d}\iota_{\alpha\pi^*(\mathcal{J}_{13}e_j)}(\mathrm{d}\pi)^\vee\pi^*(\mathcal{J}_{31}^\vee e_i)\\
&= -\alpha[\pi^*(\mathcal{J}_{13}e_i), \pi^*(\mathcal{J}_{13}e_j)] + \iota_{\alpha\pi^*(\mathcal{J}_{13}e_i)}(\mathrm{d}\pi)^\vee\pi^*\,\mathrm{d}(\mathcal{J}_{31}^\vee e_j)\\
&\quad - \iota_{\alpha\pi^*(\mathcal{J}_{13}e_j)}(\mathrm{d}\pi)^\vee\pi^*\mathrm{d}(\mathcal{J}_{31}^\vee e_i) + \frac{1}{2}\mathrm{d}\pi^*\iota_{\mathcal{J}_{13}e_i}(\mathcal{J}_{31}^\vee e_j) - \frac{1}{2}\mathrm{d}\pi^*\iota_{\mathcal{J}_{13}e_j}(\mathcal{J}_{31}^\vee e_i)\\
&= -\alpha\pi^*[\mathcal{J}_{13}e_i, \mathcal{J}_{13}e_j] + (\mathrm{d}\pi)^\vee\pi^*\iota_{\mathcal{J}_{13}e_i}\,\mathrm{d}(\mathcal{J}_{31}^\vee e_j) - (\mathrm{d}\pi)^\vee\pi^*\iota_{\mathcal{J}_{13}e_j}\,\mathrm{d}(\mathcal{J}_{31}^\vee e_i)\\
&\quad + \frac{1}{2}(\mathrm{d}\pi)^\vee\pi^*\,\mathrm{d}\iota_{\mathcal{J}_{13}e_i}(\mathcal{J}_{31}^\vee e_j) - \frac{1}{2}(\mathrm{d}\pi)^\vee\pi^*\,\mathrm{d}\iota_{\mathcal{J}_{13}e_j}(\mathcal{J}_{31}^\vee e_i)\\
&= -(\alpha + (\mathrm{d}\pi)^\vee)[(\pi^*\mathcal{J}_{13})e_i - (\pi^*\mathcal{J}_{31}^\vee)e_i, \pi^*(\mathcal{J}_{13}e_j) - \pi^*(\mathcal{J}_{31}^\vee)e_j]\\
&= -(\alpha + (\mathrm{d}\pi)^\vee)\pi^*[\mathcal{J}_{13}e_i - \mathcal{J}_{31}^\vee e_i, \mathcal{J}_{13}e_j - \mathcal{J}_{31}^\vee e_j]
\end{aligned}
$$

or

$$[a_i, a_j] = -(\alpha + (\mathrm{d}\pi)^\vee)\pi^*[\mathcal{J}_{13}e_i - \mathcal{J}_{31}^\vee e_i, \mathcal{J}_{13}e_j - \mathcal{J}_{31}^\vee e_j]. \quad (2.5)$$

Similarly, we have

$$[a_i, b_j] = (\alpha + (\mathrm{d}\pi)^\vee)\pi^*[\mathcal{J}_{13}e_i - \mathcal{J}_{31}^\vee e_i, \mathcal{J}_{22}^\vee e^j + \mathcal{J}_{12}^\vee e^j] \quad (2.6)$$

and

$$[b_i, b_j] = -(\alpha + (\mathrm{d}\pi)^\vee)\pi^*[\mathcal{J}_{22}^\vee e^i + \mathcal{J}_{12}^\vee e^i, \mathcal{J}_{22}^\vee e^j + \mathcal{J}_{12}^\vee e^j]. \quad (2.7)$$

The right-hand sides of all three of these expressions are sections of the vector bundle image$(\alpha + (\mathrm{d}\pi)^\vee) \otimes \mathbb{C}$. Therefore, the right-hand sides are sections of $E$ and in particular be sections of the graph of a map of vector bundles from image$(j + D^\vee) \otimes \mathbb{C}$ to image$(\alpha + (\mathrm{d}\pi)^\vee) \otimes \mathbb{C}$ if and only if $[a_i, a_j]$, $[a_i, a_j]$, and $[b_i, b_j]$ all vanish. This is precisely the statement of Theorem 2.3: that all pairwise Courant brackets between sections of $\mathcal{M}(\mathcal{S})$ vanish. □

Notice now that if we replace the vector bundle $V$ by $V^\vee$ and $\underline{\mathcal{J}}$ by $\underline{\hat{\mathcal{J}}}$ (see Eqs. (2.1) and (2.2)) then $\mathcal{M}$ gets replaced by

$$\hat{\mathcal{M}} = \begin{pmatrix} -\mathcal{J}_{22}^\vee & -\mathcal{J}_{13} \\ -\mathcal{J}_{12}^\vee & \mathcal{J}_{31}^\vee \end{pmatrix}, \tag{2.8}$$

but $\mathcal{M}(\mathcal{S}) = \hat{\mathcal{M}}(\mathcal{S}^\vee)$. Therefore, we have also proven Corollary 2.4.

It is also clear from this proof and using Eq. (2.4), that if $\mathcal{J}$ is integrable, then the two almost Dirac structures $\Delta = \underline{\mathcal{J}}(V) = \underline{\hat{\mathcal{J}}}(V)$ and $\hat{\Delta} = \underline{\hat{\mathcal{J}}}(V^\vee) = \underline{\mathcal{J}}(V^\vee)$ are as well. The vector bundle $\Delta$ inherits the same flat connection from $V$ via $\underline{\mathcal{J}}$ or $\underline{\hat{\mathcal{J}}}$. Similarly, $\hat{\Delta}$ inherits the same flat connection from $\underline{\mathcal{J}}$ or $\underline{\hat{\mathcal{J}}}$.

**Corollary 2.7.** *For a flat connection $\nabla$ on a vector bundle $V$ over $M$, a $\nabla$-semi-flat generalized complex structure on the total space of $V$ induces a pair of transverse Dirac structures on $M$. These Dirac structures inherit flat connections.*

**Remark 2.8.** The geometry of a pair of transversal Dirac sub-bundles was recently studied by A. Wade and found to be equivalent to a generalized paracomplex structure as defined in [37]. Furthermore, using the analysis of the integrability condition in terms of local systems above, the two Dirac structures that we have identified above form a *pair of Dirac structures* (see e.g. [12]) in the sense of Gelfand and Dorfman and therefore leads to a method of constructing integrable hierarchies with respect to the two Poisson structures coming from the two Dirac structures. This remark also applies to the torus bundle case below. Another overlap with the mathematics of integrable systems is also noted in the last section of Part I [1] and these overlaps will be the subject of future work.

Note that a *generalized Kähler structure* is defined [16] to be a generalized almost Kähler structures where both of the two generalized almost complex structures are integrable. Therefore, we have also proven (using the results of Part I, Section 4.2 [1]) the following.

**Corollary 2.9.** *The correspondence in Corollary 2.4, gives a bijective correspondence between $\nabla$-semi-flat generalized Kähler structures on $X$ and $\nabla^\vee$-semi-flat generalized Kähler structures on $\hat{X}$.*

## 3. From vector bundles to torus bundles

In this section, we describe generalized complex structures on (real) torus bundles with sections and their mirrors. The base of our torus bundles will turn out to support a pair of complimentary Dirac structures.

### 3.1. The geometry of torus bundles and the dual of a torus bundle

Let $Z \overset{p}{\to} M$ be a fiber bundle over a real manifold $M$ for which the fibers have the diffeomorphism type of a real torus of dimension $n$. We call this a *torus bundle*. So we have for $U$ small in the base, local isomorphisms of fiber bundles $p^{-1}(U) \cong U \times \boldsymbol{T}$. Assume that this fiber bundle possesses a global (smooth) section $s$. This is equivalent to assuming that the structure group of the bundle is $\mathrm{Diff}(\boldsymbol{T}, 0)$ as opposed to $\mathrm{Diff}(\boldsymbol{T})$. However, since the connected component of $\mathrm{Diff}(\boldsymbol{T}, 0)$ is contractible, the structure group of the bundle may be reduced to those diffeomorphisms which respect the group structure: $\mathrm{Aut}(\boldsymbol{T}) \cong GL(n, \mathbb{Z})$. Recently this issue was discussed in [21]. We consider this to have been done and regard $s$ as the zero section. We therefore consider $X$ as a (Lie) group bundle or bundle of (Lie) groups. Recall that for a bundle of Lie groups modeled on a Lie group $\boldsymbol{G}$ (we sometimes call this simply a $\boldsymbol{G}$-bundle) we have local maps $\rho^{-1}(U) \cong U \times \boldsymbol{G}$ and the transition maps $U \cap V \times \boldsymbol{G} \cong U \cup V \times \boldsymbol{G}$ restrict to a Lie group isomorphism of $\boldsymbol{G}$ on each fiber.

Consider the tangent sequence:

$$0 \to T_{Z/M} \to T_Z \overset{\mathrm{d}p}{\to} p^* T_M \to 0.$$

As in case of vector bundles, $T_{Z/M}$ is a pullback. Indeed, let $V = s^* T_{Z/M}$, then we have that $T_{Z/M} \cong p^* V$. This follows from the following simple observation.

**Lemma 3.1.** *Let $\boldsymbol{G}$ be a Lie group and $\mathcal{Y} \overset{p}{\to} N$ a $\boldsymbol{G}$-bundle. Call the zero-section $s$. Then we have $\rho^* s^* T_{\mathcal{Y}/N} \cong T_{\mathcal{Y}/N}$.*

**Proof.** Write any section $\sigma$ of $(s \circ \rho)^{-1} T_{\mathcal{Y}/N}$ over $U \subseteq \mathcal{Y}$ as $\sigma = \sigma_0 \circ s \circ \rho$ where $\sigma_0$ is a section of $T_{\mathcal{Y}/N}$ over $s(\rho(U))$. Now using the local group structure we may push $\sigma_0$ forward along the fibers. The transition maps respect the group structure of $\boldsymbol{G}$ and therefore these vector fields patch to a section of $T_{\mathcal{Y}/N}$ over $\rho^{-1}(\rho(U))$ and then we can restrict this section to $U$. This gives a morphism of vector bundles:

$$(s \circ \rho)^* T_{\mathcal{Y}/N} \overset{\psi}{\to} T_{\mathcal{Y}/N}.$$

Over a point $y \in \mathcal{Y}$, when we look in one of the trivial neighborhoods, $\rho^{-1}(U) \cong U \times \boldsymbol{G}$ where $y = (u, g)$, the map becomes just the obvious map $\mathrm{Lie}(\boldsymbol{G}) \to T_g \boldsymbol{G}$ which is clearly an isomorphism. Hence, we can conclude that the map $\psi$ gives an isomorphism $\rho^* s^* T_{\mathcal{Y}/N} \cong T_{\mathcal{Y}/N}$. □

Furthermore, if we use the fact that the torus *compact and connected*, the sheaf of sections, $\mathcal{V}$ of $V = s^* T_{Z/M}$ is isomorphic to $(R^1 p_* \mathbb{R})^\vee \otimes C_M^\infty$.

**Lemma 3.2.** *If $Z \overset{p}{\to} M$ is a $\boldsymbol{T} = \mathrm{Lie}(\boldsymbol{T})/\Gamma$ bundle with structure group $\mathrm{Aut}(\Gamma)$ then $\mathcal{V} \cong (R^1 p_* \mathbb{R})^\vee \otimes C_M^\infty$.*

Notice that this is just a relative version of the natural isomorphism $\mathrm{Lie}(\boldsymbol{T})^\vee \cong H^1(\boldsymbol{T}, \mathbb{R})$ which is described for example in [4].

**Proof.** Notice that $V = s^*T_{Z/M}$ is a Lie($\boldsymbol{T}$)-bundle on $M$with structure group Aut($\Gamma$). Let $\Lambda \subseteq \mathrm{tot}(V)$ be the lattice induced by $\Gamma$ and $\mathcal{S}_\Lambda$ be its sheaf of sections. There is a morphism of presheaves of abelian groups:

$$\mathcal{S}_\Lambda \to [U \mapsto H_1(p^{-1}(U), \mathbb{Z})].$$

It is given (for $U$ connected) by sending $\lambda \in \mathcal{S}_\Lambda(U)$ to the homology class of the image in $Z$ of the line in $\mathrm{tot}(V)$ connecting the point 0 over $m$ to the point $\lambda(m)$ for some $m \in U$. For $U$ small enough this is an isomorphism using the Künneth theorem:

$$\mathcal{S}_\Lambda(U) \cong H_1(p^{-1}(U), \mathbb{Z}) = \mathrm{Hom}_{\mathbb{Z}}(H^1(p^{-1}(U), \mathbb{Z}), \mathbb{Z}).$$

Hence, we have isomorphisms of sheaves:

$$\mathcal{S}_\Lambda \cong \mathrm{Hom}_{\mathbb{Z}}(R^1p_*\mathbb{Z}, \mathbb{Z}), \qquad \mathcal{S}_\Lambda \otimes \mathbb{R} \cong (R^1p_*\mathbb{R})^\vee \quad \text{and}$$
$$V \cong (R^1p_*\mathbb{R})^\vee \otimes C^\infty_M. \quad \square$$

Therefore, we have a flat Gauss–Manin connection $\nabla$ on $V$ given as the image of $1 \otimes d$ under this isomorphism. Recall that in the case of vector bundles we had for each flat connection on $V$ a (potentially) inequivalent mirror symmetry transformation. By contrast, in the case of torus bundles, the topology of a torus bundle has given us a natural flat connection on $V$ and so we need not make any additional choices.

The multisection of $V$ given by $\Lambda = \mathrm{tot}(\mathcal{S}_\Lambda)$ acts on $X = \mathrm{tot}(V)$ and the orbits are the fibers of the natural map from $X$ to $Z$. Hence, we have a diffeomorphism $X/\Lambda \cong Z$. Under this quotient, the multisection goes to $s(M)$. The sheaf of sections of $Z$ becomes the sheaf of groups $V/\mathcal{S}_\Lambda \cong ((R^1p_*\mathbb{R})^\vee \otimes C^\infty_M)/(R^1p_*\mathbb{Z})^\vee$ where the zero section has image $s(M)$. The isomorphism $X/\Lambda \cong Z$ is an isomorphism of $GL(n, \mathbb{Z})$ fiber bundles over $M$.

Now the *dual torus bundle* is defined to be $\hat{Z} = \hat{X}/\hat{\Lambda} \xrightarrow{\hat{p}} M$ where $\hat{\Lambda} = \mathrm{tot}(\mathcal{S}_{\Lambda^\vee})$ and $\mathcal{S}_{\Lambda^\vee} = \mathrm{Hom}_{\mathbb{Z}}(\mathcal{S}_\Lambda, \mathbb{Z}) \subseteq V^\vee$. Furthermore, $\mathcal{S}_{\Lambda^\vee} \cong R^1\hat{p}_*\mathbb{Z}$ and $V^\vee \cong R^1\hat{p}_*\mathbb{R} \otimes C^\infty_M$. This gives a flat connection on $V^\vee$ which is of course just the dual connection $\nabla^\vee$. Also $\mathcal{S}_{\Lambda^\vee} \otimes_{\mathbb{Z}} \mathbb{R}$ is the sheaf of flat sections of $V^\vee$ with respect to $\nabla^\vee$. The sheaf of sections of $\hat{Z}$ over $M$ is a sheaf of groups given by $V^\vee/\mathcal{S}_{\Lambda^\vee} \cong (((R^1\hat{p}_*\mathbb{R})) \otimes C^\infty_M)/(R^1\hat{p}_*\mathbb{Z})$. We then have a global section $\hat{s}$ of $\hat{Z}$ over $M$ which is the zero section and satisfies that $\hat{s}(M)$ is the image of the multisection $\hat{\Lambda}$ under the quotient map.

We saw in Part I, Section 4 [1] that if we let $X = \mathrm{tot}(V)$, then $\nabla$ gives us a splitting $D$ of the tangent sequence of the map $X \xrightarrow{\pi} M$. We can use this to split the tangent sequence of the map $Z \xrightarrow{p} M$. Consider the following diagram where we have decomposed $\pi$ as $p \circ q$.

$$X \xrightarrow{q} Z \underset{s}{\overset{p}{\rightleftarrows}} M,$$

We may push forward the exact sequence:

$$0 \to \pi^*V \to T_X \xrightarrow{d\pi} \pi^*T_M \to 0,$$

which is split by $\pi^*(V) \stackrel{D}{\leftarrow} T_X$ to the exact sequence:

$$0 \to q_*\pi^* V \to q_* T_X \stackrel{q_*\mathrm{d}\pi}{\to} q_*\pi^* T_M \to 0,$$

which is split by $q_*\pi^*(V) \leftarrow^{q_*D} q_*T_X$. Furthermore, $\Lambda$ naturally acts on all three of these sheaves and if we take the $\Lambda$ invariants of each term of this sequence we recover precisely the exact sequence that we want to split, namely

$$0 \to p^*V \to T_Z \stackrel{\mathrm{d}p}{\to} p^*T_M \to 0.$$

Therefore, the only thing to check in order to split this sequence is that the map $D$ satisfies $(\mathrm{d}t_\lambda)(Dw) = D((\mathrm{d}t_\lambda)(w))$ where for some small $U \subseteq M$ and small $U' \subseteq p^{-1}(U)$ $w$ is a section of $T_X$ over $q^{-1}(U')$, $\lambda$ is a component of $\Lambda \cap \pi^{-1}(U)$ and $t_\lambda : X \to X$ is the action of addition of $\lambda$. However,

$$\begin{aligned}(\mathrm{d}t_\lambda)(Dw) &= (\mathrm{d}t_\lambda)(((\pi^*\nabla)S)w) = ((\pi^*\nabla)(S+\lambda))((\mathrm{d}t_\lambda)w) \\ &= ((\pi^*\nabla)S)((\mathrm{d}t_\lambda)w) = D((\mathrm{d}t_\lambda)w),\end{aligned}$$

due to the fact that that the sections of the lattice are flat.

### 3.2. Generalized complex structures on torus bundles and the mirror transformation

We will now use the same names as in the vector bundles case for the splittings of the tangent sequences of $Z$ and $\hat{Z}$. That is:

$$0 \longrightarrow p^*V \underset{D}{\overset{j}{\rightleftarrows}} T_Z \underset{\alpha}{\overset{dp}{\rightleftarrows}} p^*T_M \longrightarrow 0,$$

and

$$0 \longrightarrow \hat{p}^*V^\vee \underset{\hat{D}}{\overset{\hat{j}}{\rightleftarrows}} T_{\hat{Z}} \underset{\hat{\alpha}}{\overset{d\hat{p}}{\rightleftarrows}} \hat{p}^*T_M \longrightarrow 0,$$

Since we will be using only one connection in the case of torus bundles, we will drop $\nabla$ from the notation.

**Definition 3.3.** If $M$ is an $n$-dimensional real manifold and $Z \to M$ is a real torus bundle with fiber dimension $n$ and zero section $s$ then we call a generalized almost complex structure $\mathcal{J}$ on $Z$ which comes from (see Part I, Section 4 [1]) an adapted generalized almost complex structure $\underline{\mathcal{J}}$ on $s^*T_{Z/M} \oplus T_M = V \oplus T_M$ a *semi-flat* generalized almost complex structure.

Recall that "adapted" just means that

$$\underline{\mathcal{J}}(s^*T_{Z/M} \oplus s^*T^\vee_{Z/M}) = T_M \oplus T^\vee_M.$$

As in the vector bundle case, there is a bijective correspondence between semi-flat generalized almost complex structures on $Z$ and $\hat{Z}$. The proof is precisely the same, except the choice of local coordinates is local along the base and the fiber, instead of just along the base.

**Theorem 3.4.** *A semi-flat generalized almost complex structure $\mathcal{J}$ on a torus bundle $Z \to M$ with zero section $s$ is integrable if and only if*

$$[\underline{\mathcal{J}}(\mathcal{S} \oplus \mathcal{S}^\vee), \underline{\mathcal{J}}(\mathcal{S} \oplus \mathcal{S}^\vee)] = 0,$$

*where $\mathcal{S}$ is the sheaf of flat sections of $s^*T_{Z/M}$.*

**Corollary 3.5.** *A semi-flat generalized almost complex structure $\mathcal{J}$ on a torus bundle $Z \to M$ is integrable if and only if its mirror structure $\hat{\mathcal{J}}$ on the dual torus bundle $\hat{Z} \to M$ is integrable.*

**Remark 3.6.** This means that we have given a bijective correspondence between semi-flat generalized complex structures on $Z$ and semi-flat generalized complex structures on $\hat{Z}$. The same holds true for generalized Kähler structures in which both of the generalized complex structures are semi-flat.

**Corollary 3.7.** *A semi-flat generalized almost complex structure $\mathcal{J}$ on a torus bundle $Z \to M$ induces a pair of almost Dirac structures*:

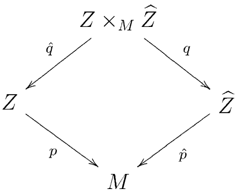
$$\Delta = \underline{\mathcal{J}}(s^*T_{Z/M}), \qquad \hat{\Delta} = \underline{\mathcal{J}}(s^*T_{Z/M}^\vee) \subseteq T_M \oplus T_M^\vee.$$

*Each carries its own flat connection and these Dirac structures are exchanged under mirror symmetry. If $\mathcal{J}$ is integrable then $\Delta$ and $\hat{\Delta}$ are integrable.*

Now in the case when the generalized complex structure on the torus bundle is of symplectic type and the torus fibers are Lagrangian this result reproduces the starting point of the work [32] where the torus bundle is written as $\mathrm{tot}(T_M^\vee)/\Lambda$ and $\Delta$ is the Dirac structure $T_M^\vee$, which inherits a flat connection $\nabla$. The mirror manifold $\mathrm{tot}(T_M)/\Lambda^\vee$ inherits a complex structure as explained in [32] constructed using the dual connection $\nabla^\vee$ which is both flat and torsion-free. This corresponds to the canonical almost complex structure on $\mathrm{tot}(T_M)$ associated to a connection on $T_M$ which is known [11] to be integrable if and only if the connection is flat and torsion-free.

## 4. The cohomology of torus bundles

Consider the diagram:
Now the space $Z \times_M \hat{Z}$ is endowed with a global closed two form given as $\Xi = \frac{1}{2\pi \mathrm{i}}\mathcal{F}$ where $\mathcal{F}$ is the curvature of a connection on the relative Poincaré (line) bundle. See [31] for an explanation of the relative Poincaré bundle in this context. Now we would like to introduce a relative version of a map given [30] in the context of mirror symmetry of abelian varieties




as introduced by Mukai [29]. This idea has appeared in various ways in [14,36,16,28] and the references therein.

**Lemma 4.1.** *If the bundle $\hat{q}^*T_{Z/M}$ is orientable then we have a morphism* (*independent of the choice of orientation*) *of sheaves of $C^\infty_M$ modules $p_*\Omega^\bullet_Z \to \hat{p}_*\Omega^\bullet_{\hat{Z}}$ is a morphism of the de Rham complexes. Therefore, this morphism gives a map of presheaves*:

$$[U \mapsto H^\bullet(p^{-1}(U), \mathbb{R})] \to [U \mapsto H^\bullet(\hat{p}^{-1}(U), \mathbb{R})].$$

*This map of presheaves induces an isomorphism of the sheafifications $R^\bullet p_*\mathbb{R} \to R^\bullet \hat{p}_*\mathbb{R}$ which decomposes into isomorphisms $R^j p_*\mathbb{R} \to R^{n-j}\hat{p}_*\mathbb{R}$ for $j = 1, \ldots, n$.*

**Proof.** We have a map $\hat{q}^\flat = (\mathrm{d}\hat{q})^\vee \circ \hat{q}^*$ from $p_*\Omega^j_Z$ to $p_*\hat{q}_*\Omega^j_{Z\times_M \hat{Z}}$ given by pulling back differential forms. Clearly, $\hat{q}^\flat$ commutes with the de Rham differentials. Observe that the map $q$ makes $Z \times_M \hat{Z}$ into a torus bundle over $\hat{Z}$. The relative tangent bundle of the tangent sequence of the map $q$ is isomorphic to $\hat{q}^*T_{Z/M}$. Therefore, we also have a map $q_*$ which integrates along the fibers and maps $q_*\Omega^k_{Z\times_M \hat{Z}}$ to $\Omega^{k-n}_{\hat{Z}}$. Explicitly, if we take our global section $s$ over $Z \times_M \hat{Z}$ of $\wedge^n \hat{q}^*T_{Z/M}$ and the corresponding global section $t$ of $\wedge^n \hat{q}^*T^\vee_{Z/M}$ then $q_*(\Upsilon) = \int_{(Z\times_M \hat{Z})/Z}((\iota_s\Upsilon) \wedge t)$. This does not depend on the choice of $s$ but we do need the fibers of $q$ to be orientable manifolds to integrate over them. Since the torus fibers of $\hat{q}$ are manifolds without boundary we have that $q_*$ and also its push-forward, $\hat{p}_*[q_*]$ commutes with the de Rham differentials. Now we can define the map $\mathbf{F}.\mathbf{T}. : p_*\Omega^\bullet_Z \to \hat{p}_*\Omega^\bullet_{\hat{Z}}$ by

$$\mathbf{F}.\mathbf{T}.(\mu) = \hat{p}_*[q_*](\hat{q}^\flat(\mu) \wedge \exp(\Xi)).$$

Now since $\mathrm{d}\Xi = 0$ we have $\mathbf{F}.\mathbf{T}.(\mathrm{d}\mu) = \mathrm{d}\mathbf{F}.\mathbf{T}.(\mu)$ and hence we get a map of presheaves $[U \mapsto H^\bullet(p^{-1}(U), \mathbb{R})] \to [U \mapsto H^\bullet(\hat{p}^{-1}(U), \mathbb{R})]$. In particular we have a natural $\mathbb{R}$-linear map $H^\bullet(Z, \mathbb{R}) \to H^\bullet(\hat{Z}, \mathbb{R})$. In order to write the map on differential forms locally on the base, chose a trivializing open neighborhood $U \subseteq M$ and $\mu \in \Omega^c(p^{-1}(U))$. Then let $\xi_i$ be the flat vertical coordinates on $p^{-1}(U)$ and $\eta_i$ be the dual flat vertical coordinates on $\hat{p}^{-1}(U)$. In these coordinates we may assume without loss of generality that

$$s = \frac{\partial}{\partial \xi_n} \wedge \cdots \wedge \frac{\partial}{\partial \xi_1}$$

on $p^{-1}(U)$. (Every section may be extended to a global section.)

Let us now express $\mu$ in local coordinates:

$$\mu = \sum_{|J|=1,\ldots,c} f_J \Theta_J \wedge \mathrm{d}\xi_{j_1} \wedge \cdots \wedge \mathrm{d}\xi_{j_b}.$$

Here, the $\Theta_J$ are pullbacks of $(c-b)$—forms from the base, $J = (j_1, \ldots, j_b)$ where $j_1 < \cdots < j_b$ and $f_J$ are functions on $p^{-1}(U)$. A simple calculation shows that

$$\hat{\mu} = \mathbf{F.T.}(\mu) = \int_{\boldsymbol{T}} \mu \wedge \exp(\mathrm{d}\xi_i \wedge \mathrm{d}\eta_i)$$

is given by

$$\hat{\mu} = \sum_{|J|=1,\ldots,c} (-1)^{k_1+\cdots+k_{n-b}} \theta_J \wedge \mathrm{d}\eta_{k_1} \wedge \cdots \wedge \mathrm{d}\eta_{k_{n-b}} \int_{\boldsymbol{T}} f_J \mathrm{d}\xi_1 \wedge \cdots \wedge \mathrm{d}\xi_n,$$

where $k_1 < \cdots < k_{n-b}$ is the compliment to $J$. Now suppose that $\mu$ is closed and that we consider the cohomology class $[\mu] \in H^j(p^{-1}(U), \mathbb{R})$, Using the Künneth isomorphism:

$$H^c(p^{-1}(U), \mathbb{R}) \cong \bigoplus_{b=0,\ldots,c} H^{c-b}(U, \mathbb{R}) \otimes H^b(\boldsymbol{T}, \mathbb{R}) \cong H^c(\boldsymbol{T}, \mathbb{R}),$$

we may absorb the $f_J$ into the $\Theta_J$ in the above expression and therefore since the cohomology of the tori $\boldsymbol{T}$ and $\boldsymbol{T}^\vee$ are generated by the classes $[\mathrm{d}\xi_1 \wedge \cdots \wedge \mathrm{d}\xi_j]$ and $[\mathrm{d}\eta_1 \wedge \cdots \wedge \mathrm{d}\eta_k]$, respectively, we conclude that the map **F.T.** induces isomorphisms $R^j p_* \mathbb{R} \to R^{n-j} \hat{p}_* \mathbb{R}$ for $j = 1, \ldots, n$ as promised. □

**Corollary 4.2.** *If $\mathcal{J}$ is a semi-flat generalized almost complex structure on a n-torus bundle with section Z on an n-manifold M with associated spinor line bundle $L \subseteq \wedge^\bullet T_Z^\vee \otimes \mathbb{C}$, and $\hat{L}$ is the line bundle associated to the mirror structure $\hat{\mathcal{J}}$ on $\hat{Z}$, then*

$$\mathbf{F.T.}(p_* L) = \hat{p}_* \hat{L} \subseteq \hat{p}_* \wedge T_{\hat{Z}}^\vee \otimes \mathbb{C}.$$

**Proof.** This follows from tensoring the previous lemma with $\mathbb{C}$ and using Lemma 6.2 of Part I [1]. □

**Remark 4.3.** The Fourier–Mukai transformation for spinors, combined with the formulae given in Lemma 6.2 of Part I [1] can easily be used to show again that integrability, phrased in terms of spinors, is preserved by the mirror transformation we have described. As we have already shown this, we do not demonstrate it again with spinors.

**Example 4.4.** Let $M = S^1$ or $\mathbb{R}$, $Z = V/\Lambda \times M$, $\hat{Z} = V^\vee/\Lambda^\vee \times M$, where $V/\Lambda \cong S^1$. Let $x, \theta, \hat{\theta}$ be "coordinates" on $M$, $V/\Lambda$ and $V^\vee/\Lambda^\vee$, respectively. Then for $f$ a complex valued smooth nowhere vanishing function on $M$:

$$\mathbf{F.T.}(\mathrm{e}^{f\,\mathrm{d}\theta\wedge\mathrm{d}x}) = \int_{V/\Lambda} \mathrm{e}^{f\,\mathrm{d}\theta\wedge\mathrm{d}x+\mathrm{d}\theta\wedge\mathrm{d}\hat{\theta}} = \int_{V/\Lambda} \mathrm{d}\theta \wedge (f\,\mathrm{d}x + \mathrm{d}\hat{\theta}) = \mathrm{d}\hat{\theta} + f\,\mathrm{d}x.$$

When we take $f = i$, we see the spinor corresponding to a symplectic structure going to one representing a complex structure.

**Remark 4.5.** Let $Z$ an $n$-torus bundle over a compact connected $n$-manifold such that $\hat{q}^* T_{Z/M}$ is orientable. Consider the "Moduli-space" $SFGCY(Z)$ of *semi-flat generalized Calabi–Yau structures*. These are semi-flat generalized complex structures which are *generalized Calabi–Yau* [18], meaning that the associated spinor line bundles $L$ have nowhere vanishing, closed, global sections.

Since these sections are known [9,16,18] to be either even or odd we may consider a "period map" [19,20] from this space into $\mathbb{P}(H^{\mathrm{even}}(Z,\mathbb{C})) \coprod \mathbb{P}(H^{\mathrm{odd}}(Z,\mathbb{C}))$. Note that we are assuming here that for a fixed structure, different closed, nowhere vanishing global sections of $L$ define the same cohomology class up to multiplication by constants. Under this assumption, we have shown the existence of a commutative diagram:

$$\begin{array}{ccc} SFGCY(Z) & \to & \mathbb{P}(H^{\mathrm{even}}(Z,\mathbb{C})) \coprod \mathbb{P}(H^{\mathrm{odd}}(Z,\mathbb{C})) \\ \downarrow & & \downarrow \\ SFGCY(\hat{Z}) & \to & \mathbb{P}(H^{\mathrm{even}}(\hat{Z},\mathbb{C})) \coprod \mathbb{P}(H^{\mathrm{odd}}(\hat{Z},\mathbb{C})) \end{array}.$$

In the case that the torus bundles are $Z$ and $\hat{Z}$ trivial, the vertical map takes horizontal $i$th cohomology to itself and vertical $i$th cohomology to vertical $(n-i)$th cohomology (both with multiplication by signs).

**Conjecture 4.6.** *Let $(Z, \mathcal{J})$ be a compact generalized Calabi–Yau manifold of (real) dimension $2n$. As we have mentioned in the previous remark, it would be desirable to know that there is a unique element in $\mathbb{P}(H^\bullet(Z,\mathbb{C}))$ associated with $Z$. Without this knowledge, the previous diagram would have to be modified by the appropriate restrictions on the left-hand side. Therefore, we conjecture that if $\phi$ is a global, closed, nowhere vanishing differential form, representing $\mathcal{J}$, and $f$ is a nowhere zero smooth complex valued function such that $\mathrm{d}(f\phi) = 0$, that $f$ is constant. If we call generalized Calabi–Yau manifolds satisfying this condition Liouville then it is easy to see that all symplectic manifolds are Liouville (take $\phi = \mathrm{e}^{-\mathrm{i}\omega}$), compact Calabi–Yau manifolds are Liouville (take $\phi$ to be a nowhere zero holomorphic $n$-form), and products and B-field transformations of Liouville generalized Calabi–Yau manifolds are Liouville.*

## 5. Examples

### 5.1. *Mirror images of B-field and β-field transforms*

Let $V$ be a vector bundle on $M$ with connection $\nabla$, $X = \mathrm{tot}(V)$ and $\mathcal{J} = F^{-1}(\pi^* \underline{\mathcal{J}})F$ a generalized almost complex structure on $X$, where $F$ is defined in Part I Section 4 [1]. We will need to relate $B$-field and $\beta$-field transforms of generalized complex structures $\mathcal{J}$ on $X$ to transformations of their mirror generalized complex structures $\hat{\mathcal{J}} = \hat{F}^{-1}(\hat{\pi}^* \underline{\hat{\mathcal{J}}})\hat{F}$ on $\hat{X}$.

The transformation $\underline{\mathcal{J}} \to \exp(\underline{B})\underline{\mathcal{J}}\exp(-\underline{B})$, where

$$\exp(\underline{B}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ B_{31} & B_{32} & 1 & 0 \\ B_{33} & B_{34} & 0 & 1 \end{pmatrix}, \quad \exp(\underline{B}) \in GL(V \oplus T_M \oplus V^\vee \oplus T_M^\vee) \tag{5.1}$$

corresponds under mirror symmetry to the transformation $\underline{\hat{\mathcal{J}}} \to \exp(\underline{\hat{B}})\underline{\hat{\mathcal{J}}}\exp(-\underline{\hat{B}})$, where

$$\exp(\underline{\hat{B}}) = \begin{pmatrix} 1 & B_{32} & B_{31} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & B_{34} & B_{33} & 1 \end{pmatrix}, \quad \exp(\underline{\hat{B}}) \in GL(V^\vee \oplus T_M \oplus V \oplus T_M^\vee). \tag{5.2}$$

Similarly, the transformation $\underline{\mathcal{J}} \to \exp(\underline{\beta})\underline{\mathcal{J}}\exp(-\underline{\beta})$, where

$$\exp(\underline{\beta}) = \begin{pmatrix} 1 & 0 & \beta_{21} & \beta_{22} \\ 0 & 1 & \beta_{23} & \beta_{24} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \exp(\underline{\beta}) \in GL(V \oplus T_M \oplus V^\vee \oplus T_M^\vee), \tag{5.3}$$

corresponds under mirror symmetry to the transformation $\underline{\hat{\mathcal{J}}} \to \exp(\underline{\hat{\beta}})\underline{\hat{\mathcal{J}}}\exp(-\underline{\hat{\beta}})$, where

$$\exp(\underline{\hat{\beta}}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \beta_{23} & 1 & 0 & \beta_{24} \\ \beta_{21} & 0 & 1 & \beta_{22} \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \exp(\underline{\hat{\beta}}) \in GL(V^\vee \oplus T_M \oplus V \oplus T_M^\vee). \tag{5.4}$$

### 5.2. *B-complex structures on* $X = \mathrm{tot}(T_M)$ *and their mirror images*

Let us examine a very simple "deformation" of the setup from [32]. It should be clear that there are many variants of this that one could easily do instead. For instance one could vary the complex structure constructed on $TM$ from a fixed choice of connection. Let $M$ be any manifold and $\nabla$ a flat and torsion-free connection on $T_M$ (one may drop the torsion free condition, but then the analysis would become more complicated). Let $X = \mathrm{tot}(T_M)$, and $\hat{X} = \mathrm{tot}(T_M^\vee)$. We will investigate $B$-field transforms of the canonical complex structure on

$X$, where $B$ is an arbitrary real two-form. We will give the condition for these transforms to be (integrable) $\nabla$-semi-flat (see Definition 2.2) generalized complex structures on $X$ and give their integrable mirror structures on $\hat{X}$.

That is to say, consider a generalized almost complex structure on $X$ of the form. The $B$-field transform of the canonical complex structure is $\mathcal{J} = F^{-1}(\pi^* \underline{\mathcal{J}})F$ where

$$\underline{\mathcal{J}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ -B_{32} - B_{33} & B_{31} - B_{34} & 0 & 1 \\ B_{31} - B_{34} & B_{32} + B_{33} & -1 & 0 \end{pmatrix} \tag{5.5}$$

and $B_{31}$ and $B_{34}$ represent arbitrary two forms on $M$, $B_{31} = -B_{31}^{\vee}$ and $B_{34} = -B_{34}^{\vee}$. For this to be semi-flat, we need it to be adapted (to the splitting, Definition 4.2 in Part I [1]) and hence $B_{32} + B_{33} = 0$. Therefore, we consider

$$\underline{\mathcal{J}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & B_{31} - B_{34} & 0 & 1 \\ B_{31} - B_{34} & 0 & -1 & 0 \end{pmatrix}. \tag{5.6}$$

Now the analysis in Section 2 tells us precisely when the generalized almost complex structure $\mathcal{J}$ is integrable. Namely, we must have that all Courant brackets of sections in the image of the subsheaf $\mathcal{S}$ of flat sections of $T_M \oplus T_M^{\vee}$ under

$$\mathcal{M} = \begin{pmatrix} -1 & 0 \\ B_{31} - B_{34} & 1 \end{pmatrix} \tag{5.7}$$

must vanish. This, in turn, is equivalent to the following three Courant brackets vanishing for any choice of flat sections $X$, $Y$ of $T_M$, and flat sections $\xi$, $\eta$ of $T_M^{\vee}$:

$$[-X + (B_{31} - B_{34})X, -Y + (B_{31} - B_{34})Y] = 0,$$
$$[-X + (B_{31} - B_{34})X, \eta] = 0, \qquad [\xi, \eta] = 0.$$

Notice that when $B = 0$, we recover no further conditions as expected. The second and third conditions are clearly vacuous. Set $B' = B_{31} - B_{34}$. The first condition then reads:

$$\iota_{-X}\,\mathrm{d}(B'Y) - \iota_{-Y}\,\mathrm{d}(B'X) - \frac{1}{2}\mathrm{d}(\iota_{-Y}(B'X) - \iota_{-X}(B'Y)) = 0$$

or

$$\begin{aligned} 0 &= -\iota_X\,\mathrm{d}\iota_Y B' - \iota_Y\,\mathrm{d}\iota_X B' + \mathrm{d}(\iota_Y \iota_X B') \\ &= -\iota_X(\mathcal{L}_Y - \iota_Y d)B' + \iota_Y(\mathcal{L}_X - \iota_X d)B' + (\mathcal{L}_Y - \iota_Y d)\iota_X B' \\ &= -\iota_{[X,Y]}B' + 2\iota_X\iota_Y\,\mathrm{d}B' + \mathcal{L}_Y\iota_X B' - \iota_Y\mathcal{L}_X B' - \iota_Y\iota_X\,\mathrm{d}B' = 3\iota_X\iota_Y\,\mathrm{d}B'. \end{aligned}$$

Thus, $\mathcal{J}$ is integrable if and only if $B' = B_{31} - B_{34}$ is closed. The mirror structure on $\hat{X}$ is given by $\hat{\mathcal{J}} = \hat{F}^{-1}(\hat{\pi}^*\underline{\hat{\mathcal{J}}})\hat{F}$ where

$$\underline{\hat{\mathcal{J}}} = \begin{pmatrix} 0 & B_{31} - B_{34} & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & B_{31} - B_{34} & 0 \end{pmatrix}. \tag{5.8}$$

This is the $\beta$-field transform of the canonical symplectic structure on $\hat{X}$, where

$$\beta = \begin{pmatrix} \hat{j} & \hat{\alpha} \end{pmatrix} \begin{pmatrix} 0 & \hat{\pi}^*(B_{31} - B_{34}) \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \hat{j}^\vee \\ \hat{\alpha}^\vee \end{pmatrix} = \hat{j}\hat{\pi}^*(B_{31} - B_{34})\hat{\alpha}^\vee.$$

### *5.3. B-symplectic structures on* $\hat{X} = \mathrm{tot}(T_M{}^\vee)$ *and their mirror transforms*

Let $\nabla^\vee$ be the dual of a flat, torsion-free connection $\nabla$ on $T_M$. In this section we will compute the conditions for a $B$-field transform of the canonical symplectic structure on $\hat{X}$ to be a $\nabla^\vee$-semi-flat generalized complex structure and find the (integrable) mirror image structure on $X$.

This $B$-symplectic generalized almost complex structure $\hat{\mathcal{J}} = \hat{F}^{-1}\underline{\hat{\mathcal{J}}}\hat{F}$ on $\hat{X}$ is given by

$$\underline{\hat{\mathcal{J}}} = \begin{pmatrix} -B_{33} & -B_{34} & 0 & 1 \\ B_{31} & B_{32} & -1 & 0 \\ B_{32}B_{31} + B_{31}B_{33} & 1 + (B_{32})^2 - B_{31}B_{34} & -B_{33}B_{31} & \\ -1 + B_{34}B_{31} - (B_{33})^2 & B_{34}B_{32} - B_{33}B_{34} & -B_{34} & B_{33} \end{pmatrix}. \tag{5.9}$$

The *adapted* requirement (see Definition 4.2 in Part I [1]) forces $B_{32} = B_{33} = 0$ and so

$$\underline{\hat{\mathcal{J}}} = \begin{pmatrix} 0 & -B_{34} & 0 & 1 \\ B_{31} & 0 & -1 & 0 \\ 0 & 1 - B_{31}B_{34} & 0 & B_{31} \\ -1 + B_{34}B_{31} & 0 & -B_{34} & 0 \end{pmatrix}. \tag{5.10}$$

Now using the above analysis on integrability we know that this the generalized almost complex structure on $\hat{X}$ will be integrable if and only if all Courant brackets of sections in the image of the subsheaf $\mathcal{S}$ of flat sections of $T_M \oplus T_M^\vee$ under

$$\mathcal{M} = \begin{pmatrix} -1 & -B_{31} \\ -B_{34} & 1 - B_{34}B_{31} \end{pmatrix} \tag{5.11}$$

must vanish. This, in turn, is equivalent to the following three Courant brackets vanishing for any choice of $X$, $Y$ flat sections of $T_M$, and $\xi$, $\eta$ flat sections of $T_M^\vee$:

$$[-X - B_{34}X, -Y - B_{34}Y] = 0, \qquad [-X - B_{34}X, -B_{31}\eta + \eta - B_{34}B_{31}\eta] = 0,$$
$$[-B_{31}\xi + \xi - B_{34}B_{31}\xi, -B_{31}\eta + \eta - B_{34}B_{31}\eta] = 0.$$

As in the previous subsection, the first equation is equivalent to $\mathrm{d}B_{34} = 0$. The second equation is equivalent to

$$[X, B_{31}\eta] = 0,$$

$$\iota_X\, \mathrm{d}\iota_{B_{31}\eta} B_{34} - \iota_{B_{31}\eta}\, \mathrm{d}\iota_X B_{34} - \frac{1}{2}\mathrm{d}(\iota_{B_{31}\eta}\iota_X B_{34} - \iota_X(B_{34}B_{31}\eta)) = 0.$$

The first of these equations simply says that $B_{31}$ is a flat bivector field. On the other hand, we claim that if $\mathrm{d}B_{34} = 0$ and $B_{31}$ is a flat bivector field then the second part of the second equation and also the third equation are also satisfied, and hence all the equations are satisfied. Indeed, we have

$$\mathrm{d}\eta = \mathrm{d}\xi = \mathrm{d}(\iota_{B_{31}\xi}\eta) = \mathrm{d}(\iota_{B_{31}\eta}\xi) = 0.$$

Therefore, the third equation gives

$$\begin{aligned}
&\iota_{B_{31}\xi}\, \mathrm{d}\iota_{B_{31}\eta} B_{34} - \iota_{B_{31}\eta}\, \mathrm{d}\iota_{B_{31}\xi} B_{34} + \frac{1}{2}(\mathrm{d}\iota_{B_{31}\xi}\iota_{B_{31}\eta} B_{34} - \mathrm{d}\iota_{B_{31}\eta}\iota_{B_{31}\xi} B_{34})\\
&\quad = \iota_{B_{31}\xi}\mathcal{L}_{B_{31}\eta} B_{34} - \iota_{B_{31}\eta}\mathcal{L}_{B_{31}\xi} B_{34} + \mathrm{d}\iota_{B_{31}\xi}\iota_{B_{31}\eta} B_{34}\\
&\quad = \iota_{B_{31}\xi}\mathcal{L}_{B_{31}\eta} B_{34} - \iota_{B_{31}\eta}\mathcal{L}_{B_{31}\xi} B_{34} - \iota_{B_{31}\xi}\, \mathrm{d}\iota_{B_{31}\eta} B_{34} + \mathcal{L}_{B_{31}\xi}\iota_{B_{31}\eta} B_{34}\\
&\quad = \iota_{[B_{31}\xi, B_{31}\eta]} B_{34} + \iota_{B_{31}\xi}\mathcal{L}_{B_{31}\eta} B_{34} - \iota_{B_{31}\xi}\, \mathrm{d}\iota_{B_{31}\eta} B_{34}\\
&\quad = \iota_{B_{31}\xi}\mathcal{L}_{B_{31}\eta} B_{34} - \iota_{B_{31}\xi}\, \mathrm{d}\iota_{B_{31}\eta} B_{34} = \iota_{B_{31}\xi}\mathcal{L}_{B_{31}\eta} B_{34} - \iota_{B_{31}\xi}\mathcal{L}_{B_{31}\eta} B_{34} = 0.
\end{aligned}$$

Similarly, the second part of the second equation gives

$$\begin{aligned}
&\iota_X\, \mathrm{d}\iota_{B_{31}\eta} B_{34} - \iota_{B_{31}\eta}\, \mathrm{d}\iota_X B_{34} - \frac{1}{2}\mathrm{d}(\iota_{B_{31}\eta}\iota_X B_{34} - \iota_X\iota_{B_{31}\eta} B_{34})\\
&\quad = \iota_X\mathcal{L}_{B_{31}\eta} B_{34} - \iota_{B_{31}\eta}\mathcal{L}_X B_{34} - \mathcal{L}_{B_{31}\eta}\iota_X B_{34} + \iota_{B_{31}\eta}\, \mathrm{d}\iota_X B_{34}\\
&\quad = \iota_{[X, B_{31}\eta]} B_{34} - \iota_{B_{31}\eta}\iota_X\, \mathrm{d}B_{34} = 0.
\end{aligned}$$

The mirror structure $\underline{\mathcal{J}}$ to $\underline{\hat{\mathcal{J}}}$ is thus given by

$$\underline{\mathcal{J}} = \begin{pmatrix} 0 & 1 - B_{31}B_{34} & 0 & B_{31} \\ -1 & 0 & B_{31} & 0 \\ 0 & -B_{34} & 0 & 1 \\ -B_{34} & 0 & -1 + B_{34}B_{31} & 0 \end{pmatrix}. \tag{5.12}$$

Therefore, the mirror $\mathcal{J}$ of $\hat{\mathcal{J}}$ is the canonical complex structure transformed by the composition of the $B$-field:

$$(\mathrm{d}\pi)^{\vee}(\pi^* B_{34})(\mathrm{d}\pi)$$

and the $\beta$-field:

$$j(\pi^* B_{31})j^{\vee}.$$

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